Trigonometric Identities – Summary

Guide to know which trigonometric function to use for integrating – look at format under radical:

not $\int \cos^5 x \sin^3 x\,dx$

Given:	Use:	
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\longrightarrow dx = a\cos\theta\,d\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\longrightarrow dx = a\sec^2\theta\,d\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\longrightarrow dx = a\sec\theta\tan\theta\,d\theta$

last topic in Section 5.7:

Integration with Partial Fractions – Introduction

Sometimes, to integrate a rational function $\frac{p(x)}{q(x)}$, it must be expressed as a sum of simpler functions.

u-sub not an option

ex. $\int \boxed{\frac{5x-4}{2x^2+x-1}}\,dx$ ← Do: factor denominator

$AC = 2 : 2\cdot 1$

$2x^2+2x-x-1$
$= 2x(x+1)-1(x+1)$
$= (x+1)(2x-1)$

↓ rewrite as sum based on factors

temporarily "lose" integral

$$\boxed{\frac{5x-4}{\cancel{2x^2+x-1}\ (x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}}$$

solve for A and B
multiply by LCD

$$\cancel{(x+1)}\cancel{(2x-1)}\frac{5x-4}{\cancel{(x+1)}\cancel{(2x-1)}} = \frac{A}{\cancel{x+1}}\cancel{(x+1)}(2x-1) + \frac{B}{\cancel{2x-1}}(x+1)\cancel{(2x-1)}$$

$5x = 2Ax + Bx$
$x(5) = x(2A+B)$

$5x - 4 = A(2x-1) + B(x+1)$

$\boxed{5x}\boxed{-4} = \boxed{2Ax}\boxed{-A}\boxed{+Bx}\boxed{+B}$

$\therefore -4 = -3 + B$
$\quad\ +3 \quad +3$
$-1 = B$

$5x = 2Ax + Bx$
$\to 5 = 2A + B$
$-5 = -2A - B$

$-4 = -A + B$
$+\ \ -5 = -2A - B$
$-9 = -3A \Rightarrow A = 3$

$u = 2x-1$, $\frac{1}{2}du = dx$; $\frac{1}{2}\int \frac{1}{u}\,du$

$$\int \frac{5x-4}{2x^2+x-1}\,dx = \int\left(\frac{3}{x+1} + \frac{-1}{2x-1}\right)dx$$

$(\ln x)' = \frac{1}{x}$

$\ln(x+1) = \frac{1}{x+1}$

$$= 3\int \boxed{\frac{1}{x+1}}\,dx - 1\int \frac{1}{2x-1}\,dx$$

$$\longrightarrow = \boxed{3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C}$$

$\frac{3}{2} = 1\frac{1}{2}$

why no u-sub? $\int \frac{5x-4}{2x^2+x-1}\,dx$

try $u = 2x^2+x-1$
$du = (4x+1)\,dx$

higher degree in numerator

$3\overline{)784}$, 2, -6, 1

$\frac{2x^3}{2x^2} = x$

1, 2, 3, 2, 0, $=1\frac{1}{2}$

If degrees of polynomials are higher in numerator, start with long division:

ex. evaluate $\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}\,dx$

$$= \int\left(x-6+\frac{5x-4}{2x^2+x-1}\right)dx$$

$\boxed{x-6}$
$\boxed{2x^2+x-1}\overline{)\cancel{2x^3}-11x^2-2x+2}$
$+\ \ -\cancel{2x^3}-x^2+x$
$\quad -\cancel{12x^2}-x+2$
$+\ \ \cancel{12x^2}+6x-6$
$\quad 5x-4$

$x(2x^2+x-1)$
$-(2x^3+x^2-x)$
$= -2x^3-x^2+x$

$$\begin{array}{r} -12x^2 - x + 2 \\ + \quad 12x^2 + 6x - 6 \\ \hline \boxed{5x-4} \end{array}$$

$= -2x^3 - x^2 + x$

$-6(2x^2+x-1)$
$= (-12x^2 - 6x + 6)$
$= 12x^2 + 6x - 6$

$$= \int \left(x - 6 + \frac{5x-4}{2x^2+x-1}\right) dx$$

$$= \int (x-6)\,dx + \underbrace{\int \left(\frac{5x-4}{2x^2+x-1}\right) dx}$$

$$= \frac{x^2}{2} - 6x + 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C$$

ex. $\displaystyle\int \frac{x-5}{\boxed{2x^3+7x^2-4x}}\,dx \xrightarrow{\text{factor}} x(2x^2+7x-4)$

$= x(2x-1)(x+4)$

$$\left(\frac{x-5}{x(2x-1)(x+4)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+4}\right) x(2x-1)(x+4)$$

$$x - 5 = A(2x-1)(x+4) + Bx(x+4) + Cx(2x-1)$$

$$= A(2x^2+7x-4) + Bx^2 + 4Bx + 2Cx^2 - Cx$$

$$0x^2 + x - 5 = 2Ax^2 + 7Ax - 4A + Bx^2 + 4Bx + 2Cx^2 - Cx$$

$0 = 2A + B + 2C \qquad 1 = 7A + 4B - C \qquad -5 = -4A \Rightarrow A = \frac{5}{4}$

$0 = 2 \cdot \frac{5}{4} + B + 2C \qquad 1 = 7 \cdot \frac{5}{4} + 4B - C$

$0 = \frac{5}{2} + B + 2C \qquad 1 = \frac{35}{4} + 4B - C$

$\boxed{-\frac{5}{2} = B + 2C \qquad -\frac{31}{4} = 4B - C}$

updated ~dk

$$\boxed{\begin{aligned} A &= \tfrac{5}{4} \\ B &= -2 \\ C &= -\tfrac{1}{4} \end{aligned}}$$

$$\frac{5}{4}\int \frac{1}{x}\,dx - 2\int \frac{1}{2x-1}\,dx + \frac{9}{4}\int \frac{1}{x+4}\,dx$$

$$= \frac{5}{4}\ln|x| - \ln|2x-1| - \frac{1}{4}\cdot\ln|x+4| + C$$

ex. Expand to 3 terms:

$$\frac{2x^2+9x+8}{(x+2)^2(x+1)} \xrightarrow{\text{EXPAND}} \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+1}$$

Recall: $(\arctan x)' = \dfrac{1}{1+x^2}$

then $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

ex. $\displaystyle\int \frac{dx}{4+x^2} = \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C} \leftarrow$ check by differentiating

$a = 2$

evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3+4x}\,dx \longrightarrow x(x^2+4)$

doesn't reduce/factor past quadratic

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Improper Integrals - Introduction

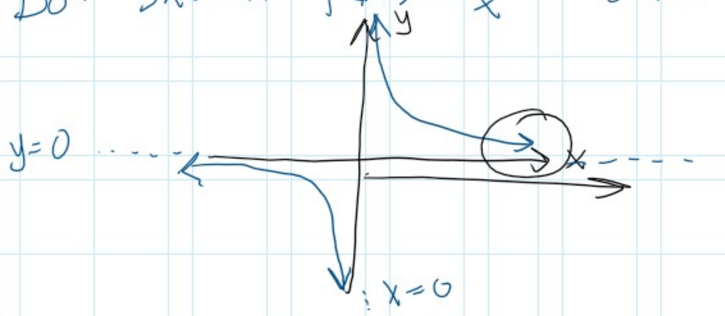
Integral is considered improper if:

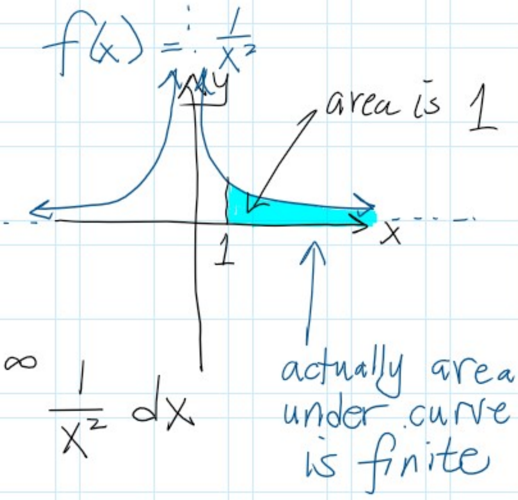
- one or both of the bounds are infinity ,

Integral is considered improper if:
- one or both of the bounds are infinity
- integrand is discontinuous in the interval

$\int_a^b \underbrace{f(x)}_{\text{integrand}} dx$

Do: sketch $f(x)=\frac{1}{x}$ then $f(x)=\frac{1}{x^2}$





ex. $\int_1^{\infty} \frac{1}{x^2}\,dx$

$$\int_1^2 \frac{1}{x^2}dx = \int_1^2 x^{-2}dx$$
$$= -x^{-1}\Big|_1^2 \qquad F(b)-F(a)$$
$$= -\frac{1}{x}\Big|_1^2 = -\left(\frac{1}{2}-1\right) = -\frac{1}{2}+1$$

$$\int_1^3 \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^3 = -\left(\frac{1}{3}-1\right) = -\frac{1}{3}+1$$

$$\int_1^{100} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{100} = -\left(\frac{1}{100}-1\right) = \underbrace{-\frac{1}{100}}+1$$

$$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{t\to\infty} \underbrace{\int_1^t \frac{1}{x^2}dx}$$ $\qquad \frac{1}{\text{lg } \#} = 0$
$$= \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$
$$= \lim_{t\to\infty} -\left(\frac{1}{t}-1\right)$$
$$= \underbrace{\lim_{t\to\infty}\left(-\frac{1}{t}+1\right)} = \cancel{-\lim_{t\to\infty}\frac{1}{t}}^{\,0} + 1 = \boxed{1}$$

Definition:

if $\int_a^t f(x)dx$ exists for every number $t \geq a$ then

constant

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

if $\int_t^b f(x)dx$ exists for every number $t \leq b$ then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

this type of integral is called:
- CONVERGENT if corresponding limit exists
- DIVERGENT " DNE

DIVERGENT DNE

assuming both scenarios are convergent then:

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\text{converges}} + \underbrace{\int_{a}^{\infty} f(x)\,dx}_{\text{converges}}$$

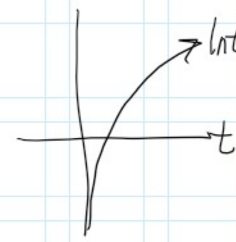
ex. consider $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \left(\ln|t| - \cancel{\ln 1}\right)$

$= \infty$ OR DNE

DIVERGES



use IBP

LIATE

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\int u\,dv = uv - \int v\,du$

ex. evaluate $\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$

$= \lim_{t\to-\infty}\left(xe^x\Big|_t^0 - \int_t^0 e^x\,dx\right)$

$= \lim_{t\to-\infty}\left(0 - te^t - e^x\Big|_t^0\right)$

$= \lim_{t\to-\infty}\left(-te^t - (e^0 - e^t)\right)$

$= \lim_{t\to-\infty}\left(-te^t - 1 + e^t\right)$

$= -\lim_{t\to-\infty} \underset{-\infty\cdot 0}{te^t} - 1 + \cancel{\lim_{t\to-\infty} e^t}$

indeterminate

rewrite as a fraction

$= -\lim_{t\to\infty} \frac{t}{e^{-t}} \quad \boxed{\frac{-\infty}{\infty}}$

$\xrightarrow{L'H} -\lim_{t\to-\infty} \frac{1}{-e^{-t}}$

$= \cancel{\lim_{t\to-\infty} e^t}$

-1

-1

$\boxed{-1}$

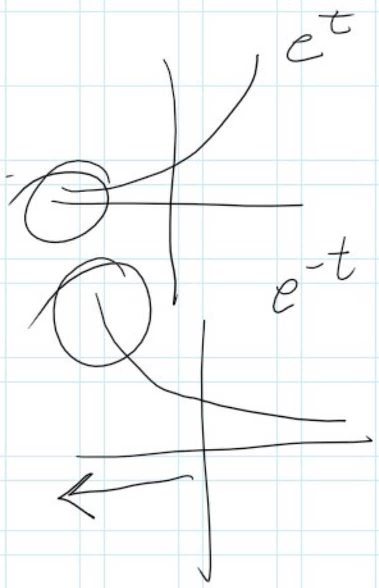
Indeterminate Forms

$\frac{0}{0}$

$\frac{\infty}{\infty}$

$0\cdot\infty$

0^{∞}



$\frac{1}{x^{-5}} = x^5$

ex. evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$

DO:

$= \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx$

$= \lim_{t\to\infty} \arctan x \Big|_0^t$

$= \lim_{t\to\infty} \arctan t - 0$

$= \pi/2$

$\tan 0 = 0$

$\therefore \arctan 0 = 0$

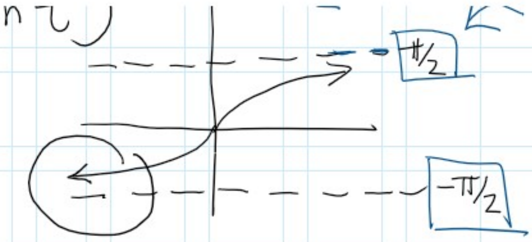
$= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx$

$= \lim_{t\to-\infty} \arctan x \Big|_t^0$

$= \lim_{t\to-\infty}\left(\cancel{\arctan 0} - \arctan t\right)$

$= \lim_{t \to -\infty} \left(\arctan 0 - \arctan t \right)$

$= -\underbrace{\lim_{t \to -\infty} \arctan t}_{-(-\pi/2)} = \frac{\pi}{2}$



ex. for what values of p is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

already know: $\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\int_1^\infty \frac{1}{x}\,dx$ diverges

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ converges when } p>1, \text{ diverges when } p \le 1}$$

Improper Integrals with Discontinuity

if f is continuous on $[a, b)$ $\quad \int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$

discontinuity @ b

if f is continuous on $(a, b]$ $\quad \int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$

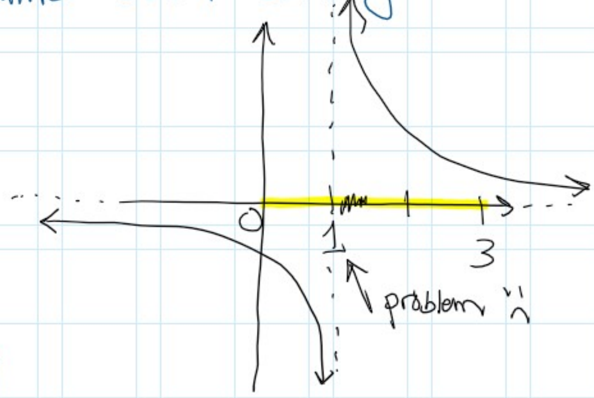
discontinuity @ a

if f has a discontinuity at c where $a<c<b$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

assume both convergent

ex. evaluate $\int_0^3 \frac{1}{x-1}\,dx$, if possible



$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$

$= \lim_{t \to 1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{t \to 1^+} \int_t^3 \frac{1}{x-1}\,dx$

$= \underbrace{\lim_{t \to 1^-} \ln|x-1| \Big|_0^t}_{} + \boxed{\lim_{t \to 1^+} \ln|x-1| \Big|_t^3}$

1

$= \underbrace{\lim_{t \to 1^-} \left(\ln|t-1| \right.}_{} - \left. \cancel{\ln|-1|} \right)$

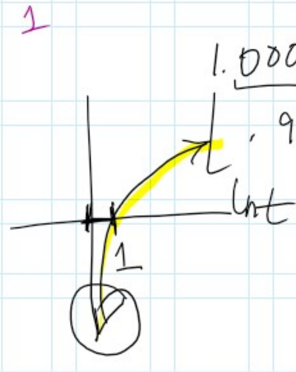
really sm+ #

$\ln 1$

0

$\boxed{-\infty}$

$\boxed{\text{DIVERGES}}$



$|.9999 - 1| =$ really sm positive #

ex. $\int_0^1 \ln x\,dx = \lim_{t \to 0^+} \int_t^1 \ln x\,dx \leftarrow$ IBP or $\ln x^n$ reduction formula

discontinuity @ x=0

$= \lim_{t \to 0^+} \left(x \ln x - x \right) \Big|_t^1$

$= \lim \left(\cancel{1 \ln 1}^{\,0} - 1 - (t \ln t - t) \right)$

$$= \lim_{t\to 0^+} \left(\cancel{1\ln 1}^{\to 0} - 1 - (t\ln t - t) \right)$$

$$= -1 - \lim_{t\to 0^+} t\ln t + \cancel{\lim_{t\to 0^+} t}^{\to 0}$$

$0 \cdot -\infty$

indeterminate

$$t = \frac{1}{\frac{1}{t}}$$

$$= -1 - \underbrace{\lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}}_{\downarrow L'H}$$

$$\frac{1}{t} \cdot \frac{t^2}{1} = t$$

$$= -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= \boxed{-1} + \lim_{t\to 0^+} t \to 0$$